TUTORIAL I (LINEAR ALGEBRA)

MS 103: Mathematics II

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1. Prove that the set of all $m \times n$ matrices with entries from a field $\mathbb{F}$, denoted by $M_{m\times n}$ with the following operations of matrix addition and scaler multiplication: for $A,\ B \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,
$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} \ and \ (cA)_{ij} = c(A)_{ij}.$$

2. Let S be any non-empty set and $\mathbb{F}$ be any field. Prove that the set of all functions from S to $\mathbb{F}$, denoted by $\mathcal{F}(S,\mathbb{F})$ is a vector space with the following operations of addition and scaler multiplication defined for $f, g \in \mathcal{F}(S,\mathbb{F})$ and $c \in \mathbb{F}$ by
$$(f+g)(s) = f(s) + g(s) \ and \ (cf)(s) = cf(s)$$
for each $s \in S$.

3. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinatewise, and for $c \in \mathbb{F}$ and $(a_1, a_2) \in V$, define
$$c(a_1, a_2) = (a_1, 0).$$
Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

4. Let V and W be vector spaces over a field $\mathbb{F}$. Let
$$Z = \{(v, w) : v \in V \ and \ w \in W\}.$$
Prove that Z is a vector space over $\mathbb{F}$ with the operations
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \ and \ c(v_1, w_1) = (cv_1, cw_1).$$

5. For each of the following list of vectors, determine whether the first vector can be expressed as a linear combination of the other two:
   (1) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$ in $\mathbb{R}^3$
   (2) $(3, 4, 1), (1, -2, 1), (-2, -1, 1)$ in $\mathbb{R}^3$
   (3) $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$ in $\mathbb{R}^3$
   (4) $x^3 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1$ in $P_3(\mathbb{R})$.
   (5) $6x^3 - 3x^2 + x + 2, x^3 - x^2 + 2x + 3, 2x^3 - 3x + 1$ in $P_3(\mathbb{R})$.

6. Determine whether the given vectors is in the span of S:
   (1) $(-1, 1, 1, 2)$, $S = \{(1, 0, 1, -1), (0, 1, 1, 1)\}$
   (2) $-x^3 + 2x^2 + 3x + 3$, $S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$
   (3) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
   (4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

7. Show that if $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

8. Show that the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ generates $M_{2\times 2}(\mathbb{F})$.

9. Show that the polynomials $x^2 + 3x - 2,\ 2x^2 + 5x - 3$ and $-x^2 - 4x + 4$ generated $P_2(\mathbb{R})$.

10. Show that in $M_{2\times 3}(\mathbb{R})$, the set $\left\{ \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} \right\}$ is linearly dependent.

11. Prove that the set $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ is linearly independent.

12. Determine whether the following sets are linearly dependent or linearly independent
   (1) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$.
   (2) $\{(1,-1,2),(1,-2,1),(1,1,4)\}$ in $\mathbb{R}^3$.
   (3) $\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0, \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$.
   (4) $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x\}$ in $P_4(\mathbb{R})$.

13. Let V be a vector space and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent then prove that S_2 is also linearly dependent.

14. Let V be a vector space and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent then prove that S_1 is also linearly independent.

15. In $M_{m\times n}(\mathbb{F})$, let E_{ij} denote the matrix whose only non-zero entry is a 1 in the i-th row and j-th column. Then prove that $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m\times n}(\mathbb{F})$.

16. Prove that $\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$ is a basis for $P_2(\mathbb{R})$.

17. Determine whether the following sets are basis for the given vector spaces, justify your answer.
   (1) $\{(1,2,-1),(1,0,2),(2,1,1)\}$ in $\mathbb{R}^3(\mathbb{R})$.
   (2) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$ in $\mathbb{R}^3(\mathbb{R})$.
   (3) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$ in $P_2(\mathbb{R})$.
   (4) $\{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$ in $P_2(\mathbb{R})$.

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